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ON THE ORBIT OF HYPERION.

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1. The principal difficulty in the integration of the equations of motion in the case of the problem of three bodies arises in the integration of terms involving the inverse powers of the distance between the disturbed and disturbing bodies. When the ratio between the radius vectors is not too near unity, the inverse powers referred to can be developed in rapidly converging series, in terms of multiples of the elongation. When, however, these ratios do not differ greatly from unity, the convergence of the series mentioned is very slow. If, in addition, the mean motions of the two bodies are nearly commensurate, the ordinary methods of solving the problem become inapplicable.

Such a case presents itself in the determination of the perturbations of Hyperion produced by Titan. On the other hand, the mutual inclination of the orbits of these two satellites is so small as to have eluded detection; the eccentricity of the orbit of Titan is less than 0.03; and the position of the apo-saturnium of Hyperion so nearly coincides (at least at present) with the point of conjunction of the two satellites, as to give rise to a suspicion that the eccentricity of its orbit is small, and that the apparent eccentricity is principally due to the perturbations produced by Titan.

2. *First Approximation.*— These circumstances have suggested* the propriety of first investigating that part of the disturbance which may be determined by neglecting the mutual inclination and the eccentricities of the orbits of both bodies. On this hypothesis Tisserand has found that, assuming the mass of Titan $m' = \frac{1}{10760}$, the motion of Hyperion may be represented, to terms of the first order with respect to the mass, by the formulæ,

$$\frac{r}{a} = 1 - 0.0004 \cos (l' - l) - 0.0014 \cos 2(l' - l) + 0.1000 \cos 3(l' - l) + 0.0006 \cos 4(l' - l),$$

*Tisserand. Sur un Cas Remarquable du Problème des Perturbations. Comptes-Rendus, 1886, 2e Semestre, p. 446.

Hill. Coplanar Motion of Two Planets, One Having a Zero Mass. ANNALS OF MATHEMATICS. Vol. III. p. 65.

$$w = l + 10' \sin (l' - l) + 13' \sin 2(l' - l) - 683' \sin 3(l' - l) - 3' \sin 4(l' - l), \quad (1)$$

where r is the radius vector of its orbit,

a mean distance from Saturn,

w true longitude in orbit,

l mean longitude in orbit,

l' mean longitude in orbit of Titan.

3. *Second Approximation.*—As a second approximation let us assume

$$r = a(1 + a_1 \cos\theta + a_2 \cos 2\theta + \ldots) = a(1 + \sigma), \quad (2)$$

$$\frac{dw}{dt} = n(1 + n_1 \cos\theta + n_2 \cos 2\theta + \ldots) = n(1 + \tau), \quad (3)$$

where r and w are the radius vector and longitude in orbit of Hyperion, $\theta = l' - l = (n' - n)t$ is the mean angular distance between the radius vectors of Titan and Hyperion, a and n are the mean values of r and w, and a_1, n_1, etc. are constants to be determined.

The differential equations of motion may be written

$$r^2 \frac{dw}{dt} = m'k^2 \int S r dt + C, \quad (4)$$

$$\frac{d^2r}{dt^2} - r\left(\frac{dw}{dt}\right)^2 + \frac{k^2}{r^2} = m'k^2 R, \quad (5)$$

where k is the constant of the system as derived from the orbit of Titan, m' the mass of Titan, $m'k^2R$ and $m'k^2S$ the components in the plane of the orbit of the disturbing force in the direction of and perpendicular to the disturbed radius vector, and C is a constant of integration. As is well known, when the orbit is nearly circular, C is approximately equal to $k\sqrt{a}$; say $k\sqrt{[a(1-\nu)]}$, where ν is a small quantity.

4. Substituting in (4) the values of r and dw/dt given by (2) and (3), and dividing by a^2n, we have

$$\frac{r^2}{a^2 n}\frac{dw}{dt} = 1 + 2\sigma + \tau + \sigma^2 + 2\sigma\tau + \sigma^2\tau = \frac{k\sqrt{(1-\nu)}}{a^2 n} + m'\frac{k^2}{a^2 n}\int S r dt, \quad (6)$$

in which, if $\Sigma = \sum_{i=1}^{i=\infty} \Sigma_i$, i being integral and positive,

$$\sigma = a_1 \cos\theta + a_2 \cos 2\theta + \ldots = \Sigma a_i \cos i\theta,$$

$$\tau = n_1 \cos\theta + n_2 \cos 2\theta + \ldots = \Sigma n_i \cos i\theta,$$

$$\sigma^2 = \tfrac{1}{2}\Sigma a_i^2 + \Sigma a_i a_{i+1} \cos\theta + (\tfrac{1}{2}a_1^2 + \Sigma a_i a_{i+2}) \cos 2\theta$$
$$+ (a_1 a_2 + \Sigma a_i a_{i+3}) \cos 3\theta + (\tfrac{1}{2}a_2^2 + a_1 a_3 + \Sigma a_i a_{i+4}) \cos 4\theta$$
$$+ (a_1 a_4 + a_2 a_3 + \Sigma a_i a_{i+5}) \cos 5\theta + \ldots$$

5. Comparing (2) and (3) with Tisserand's values of r/a and w, we are led to infer that it will be proper, for the present, to consider a_3 and n_3 as quantities of the first order; a_1, a_2, a_4, n_1, n_2, n_4, as quantities of the third order; a_5, a_6, a_7, a_8, n_5, n_6, n_7, n_8 as quantities of the fourth order; and neglect the remaining coefficients of σ and τ.

We have, then, to terms of the fourth order,

$$\sigma = a_1 \cos\theta + a_2 \cos 2\theta + \ldots + a_8 \cos 8\theta, \tag{7}$$
$$\tau = n_1 \cos\theta + n_2 \cos 2\theta + \ldots + n_8 \cos 8\theta,$$
$$\sigma^2 = \tfrac{1}{2}a_3^2 + a_3(a_2 + a_4) \cos\theta + a_1 a_3 \cos 2\theta + a_1 a_3 \cos 4\theta \tag{8}$$
$$+ a_2 a_3 \cos 5\theta + \tfrac{1}{2}a_3^2 \cos 6\theta + a_3 a_4 \cos 7\theta,$$
$$2\sigma\tau = a_3 n_3 + [(a_2 + a_4) n_3 + a_3(n_2 + n_4)] \cos\theta$$
$$+ (a_1 n_3 + a_3 n_1) \cos 2\theta + (a_1 n_3 + a_3 n_1) \cos 4\theta$$
$$+ (a_2 n_3 + a_3 n_2) \cos 5\theta + a_3 n_3 \cos 6\theta$$
$$+ (a_3 n_4 + a_4 n_3) \cos 7\theta,$$
$$\sigma^2\tau = \tfrac{3}{4}a_3^2 n_3 \cos 3\theta + \tfrac{1}{4}a_3^2 n_3 \cos 9\theta.$$

Whence (6) becomes

$$\frac{r^2}{a^2 n}\frac{dw}{dt} = 1 + \tfrac{1}{2}a_3^2 + a_3 n_3$$
$$+ [2a_1 + n_1 + a_3(a_2 + a_4 + n_2 + n_4) + (a_2 + a_4) n_3] \cos\theta$$
$$+ [2a_2 + n_2 + a_3(a_1 + n_1) + a_1 n_3] \cos 2\theta$$
$$+ [2a_3 + n_3 + \tfrac{3}{4}a_3^2 n_3] \cos 3\theta$$
$$+ [2a_4 + n_4 + a_3(a_1 + n_1) + a_1 n_3] \cos 4\theta$$
$$+ [2a_5 + n_5 + a_3(a_2 + n_2) + a_2 n_3] \cos 5\theta$$
$$+ [2a_6 + n_6 + \tfrac{1}{2}a_3^2 + a_3 n_3] \cos 6\theta$$
$$+ [2a_7 + n_7 + a_3(a_4 + n_4) + a_4 n_3] \cos 7\theta$$
$$+ [2a_8 + n_8] \cos 8\theta$$
$$= \frac{k\sqrt{(1-\nu)}}{a^3 n} + m'\frac{k^2}{a^2 n}\int S r dt. \tag{9}$$

6. Remembering that $C = k\sqrt{[a(1-\nu)]}$, equation (4) also gives

$$r\left(\frac{dw}{dt}\right)^2 = \frac{1}{r^3}\left(r^2\frac{dw}{dt}\right)^2$$

$$= \frac{k^2}{r^3}\left(m'k\int Srdt + \sqrt{[a(1-\nu)]}\right)^2$$

$$= \frac{k^2}{r^3}a(1-\nu) + m'\frac{k^2}{r^3}\left[k\sqrt{[a(1-\nu)]}\,.\,2\int Srdt + m'k^2\left(\int Srdt\right)^2\right];$$

whence (5) becomes $\dfrac{d^2r}{dt^2} + \dfrac{k^2}{r^3}[r - a(1-\nu)] = m'k^2P,$

where $$P = R + \frac{1}{r^3}\left[k\sqrt{[a(1-\nu)]}\,.\,2\int Srdt + m'k^2\left(\int Srdt\right)^2\right]; \quad (10)$$

or since $r = a(1+\sigma)$, $$\frac{d^2\sigma}{dt^2} + \frac{k^2}{a^3}\frac{\sigma+\nu}{(1+\sigma)^3} = m'\frac{k^2}{a}P. \quad (11)$$

7. Differentiating twice with regard to t, since $\theta = (n'-n)t$, where n' is the mean motion of Titan, this gives

$$\frac{d^2\sigma}{dt^2} = -(n'-n)^2(a_1\cos\theta + 4a_2\cos 2\theta + 9a_3\cos 3\theta + \ldots). \quad (12)$$

Expanding by the binomial theorem,

$$\frac{\sigma+\nu}{(1+\sigma)^3} = \nu + (1-3\nu)\sigma - 3(1-2\nu)\sigma^2 + 2(3-5\nu)\sigma^3 - 5(2-3\nu)\sigma^4$$
$$+ 3(5-7\nu)\sigma^5 - 7(3-4\nu)\sigma^6 + \ldots.$$

In the development of the third and higher powers of σ, it will be sufficient to put $\sigma = a_3\cos 3\theta$; whence

$$\begin{aligned}
\sigma^3 &= \tfrac{3}{4}a_3^3\cos 3\theta + \tfrac{1}{4}a_3^3\cos 9\theta,\\
\sigma^4 &= \tfrac{3}{8}a_3^4 + \tfrac{1}{2}a_3^4\cos 6\theta + \tfrac{1}{8}a_3^4\cos 12\theta,\\
\sigma^5 &= \tfrac{5}{8}a_3^5\cos 3\theta + \tfrac{5}{16}a_3^5\cos 9\theta + \tfrac{1}{16}a_3^5\cos 15\theta,\\
\sigma^6 &= \tfrac{5}{16}a_3^6 + \tfrac{15}{32}a_3^6\cos 6\theta + \tfrac{3}{16}a_3^6\cos 12\theta + \tfrac{1}{32}a_3^6\cos 18\theta.
\end{aligned} \quad (13)$$

8. If we neglect terms whose arguments are multiples of θ greater than 8θ, we have to terms of the fourth order

$$\frac{\sigma+\nu}{(1+\sigma)^3} = \nu - \tfrac{3}{2}a_3^2(1-2\nu) - \tfrac{15}{8}a_3^4(2-3\nu)$$

$$
\begin{aligned}
&+ [a_1(1-3\nu) - 3a_3(a_2+a_4)(1-2\nu)]\cos\theta \\
&+ [a_2(1-3\nu) - 3a_1a_3(1-2\nu)]\cos 2\theta \\
&+ [a_3(1-3\nu) + \tfrac{3}{2}a_3^3(3-5\nu)]\cos 3\theta \\
&+ [a_4(1-3\nu) - 3a_1a_3(1-2\nu)]\cos 4\theta \qquad (14)\\
&+ [a_5(1-3\nu) - 3a_2a_3(1-2\nu)]\cos 5\theta \\
&+ [a_6(1-3\nu) - \tfrac{3}{2}a_3^2(1-2\nu) - \tfrac{5}{2}a_3^4(2-3\nu)]\cos 6\theta \\
&+ [a_7(1-3\nu) - 3a_3a_4(1-2\nu)]\cos 7\theta \\
&+ a_8(1-3\nu)\cos 8\theta.
\end{aligned}
$$

Since the left hand members of (9) and (11) contain only terms involving cosines of multiples of θ and constant coefficients it is evident that the right hand members of the same contain similar terms only, and we may put

$$\frac{k^2}{a^2n(n'-n)}\int Srd\theta = \sum_1^\infty S_i\cos i\theta,$$

$$a^2P = \sum_0^\infty P_i\cos i\theta;$$

whence equations (9) and (11) may be written in the form

$$\frac{r^2}{a^2n}\frac{dw}{dt} = \frac{k\sqrt{(1-\nu)}}{a^3n} + m'\sum_1^\infty S_i\cos i\,\theta, \qquad (15)$$

$$\frac{a^3}{k^2}\frac{d^2\sigma}{dt^2} + \frac{\sigma+\nu}{(1+\sigma)^3} = m'\sum_0^\infty P_i\cos i\theta. \qquad (16)$$

9. With assumed values of our constants, R and S are computed for particular values of θ, selected at equal intervals from $\theta = 0$ to $\theta = 180°$, and then expanded mechanically into a series of cosines of multiples of θ, effected with constant coefficients; whence P and $\int Srd\theta$, and thence P_i and S_i are easily obtained.

If now we replace the left hand members of equations (15) and (16) by the developed expressions of (9) and (11), and equate the coefficients of the cosines of equal multiples of θ, there will result any number of equations from which may be derived a corresponding number of the quantities a_1, n_1, etc. Instead, however, of considering the mass of Titan as known, it will be better to assume a_3 as given, and to consider m' as one of the unknowns.

With Tisserand's values of the coefficients, the constant part of (16) gives an approximate value of ν; with this value of ν the constant part of (15) gives an approximate value of $\mu = a^3(n'-n)^2/k^2$; with these values of μ and ν, the terms of (16) involving 3θ give an approximate value of m'. On account of the impor-

tance of these quantities I have developed the equations giving μ, m', and ν to terms of the sixth, seventh, and eighth orders respectively. The remaining equations have been carried only to terms of the fourth order.

10. The equations giving ν, μ, and m' are as follows, including terms of the orders named:—

$$\nu\left[1+3(a_1^2+a_2^2+a_3^2+a_4^2)+\tfrac{45}{8}a_3^4+\tfrac{35}{4}a_3^6+\tfrac{1575}{128}a_3^8\right] \qquad (17)$$
$$=\tfrac{3}{2}(a_1^2+a_2^2+a_3^2+a_4^2)+\tfrac{15}{4}a_3^4+\tfrac{105}{16}a_3^6+\tfrac{315}{32}a_3^8+m'P_0,$$

$$\mu\left\{\begin{array}{l}1+\frac{1}{2}(a_1^2+a_2^2+a_3^2+a_4^2)\\ \quad+a_1n_1+a_2n_2+a_3n_3+a_4n_4+\frac{1}{4}a_3^2n_6\end{array}\right\}^2=\left(\frac{n'-n}{n}\right)^2(1-\nu), \qquad (18)$$

$$\begin{aligned}m'P_3=&(1-3\nu-9\mu)a_3-3(1-2\nu)a_1(a_2+a_4)\\&+\tfrac{9}{2}(1-\tfrac{5}{3}\nu)a_3^3+\tfrac{75}{8}(1-\tfrac{7}{5}\nu)a_3^5+\tfrac{2475}{128}(1-\tfrac{9}{7}\nu)a_3^7.\end{aligned} \qquad (19)$$

Comparing (16) with (12) and (14) the coefficients of $\cos\theta$, $\cos 2\theta$, $\cos 4\theta$ give the following equations for the determination of a_1, a_2, a_3:—

$$\begin{aligned}(1-3\nu-\mu)a_1-3a_3(1-2\nu)a_2-3a_3(1-2\nu)a_4&=m'P_1,\\-3a_3(1-2\nu)a_1+(1-3\nu-4\mu)a_2&=m'P_2,\\-3a_3(1-2\nu)a_1+(1-3\nu-16\mu)a_4&=m'P_4.\end{aligned} \qquad (20)$$

Similarly, for $a_5, \ldots a_8$, we have

$$\begin{aligned}(1-3\nu-25\mu)a_5&=m'P_5+3a_3(1-2\nu)a_2,\\(1-3\nu-36\mu)a_6&=m'P_6+3a_3(1-2\nu)\cdot\tfrac{1}{2}a_3+5a_3^4(1-\tfrac{3}{2}\nu),\\(1-3\nu-49\mu)a_7&=m'P_7+3a_3(1-2\nu)a_4,\\(1-3\nu-64\mu)a_8&=m'P_8.\end{aligned} \qquad (21)$$

In the same way n_3 is obtained from

$$(1+\tfrac{3}{4}a_3^2)n_3=m'S_3-2a_3, \qquad (22)$$

which is derived by comparing (15) with (9); after which n_1, n_2, and n_4 are obtained from

$$\begin{aligned}n_1+a_3n_2+a_3n_4&=m'S_1-2a_1-(a_3+n_3)(a_2+a_4),\\a_3n_1+n_2&=m'S_2-2a_2-(a_3+n_3)a_1,\\a_3n_1+n_4&=m'S_4-2a_4-(a_3+n_3)a_1;\end{aligned} \qquad (23)$$

and the remaining terms give

$$\begin{aligned} n_5 &= m'S_5 - 2a_5 - (a_3 + n_3)\,a_2 - a_3 n_2, \\ n_6 &= m'S_6 - 2a_6 - \tfrac{1}{2}a_3^2 - a_3 n_3, \\ n_7 &= m'S_7 - 2a_7 - a_3\,(a_4 + n_4), \\ n_8 &= m'S_8 - 2a_8. \end{aligned} \qquad (24)$$

With the values of the constants thus obtained, new values are determined. This process must be repeated until the values obtained agree with those assumed.

11. For the determination of R and S were employed the following: —

$$n = 16^\circ.9199, \qquad n' = 22^\circ.5770,$$

$$\frac{r}{a} = 1 - 0.0004 \cos\theta - 0.0014 \cos 2\theta + 0.1000 \cos 3\theta + 0.0006 \cos 4\theta,$$

$$w' - w = \theta - 10' \sin\theta - 13' \sin 2\theta + 683' \sin 3\theta + 3' \sin 4\theta,$$

$$\log a = 0.0835122, \qquad r' = 1,$$

$$\rho^2 = r'^2 \sin^2(w' - w) + [r' \cos(w' - w) - r]^2,$$

$$h = \frac{1}{\rho^3} - \frac{1}{r'^3},$$

$$R = hr' \cos(w' - w) - \frac{r}{\rho^3},$$

$$S = hr' \sin(w' - w).$$

With these data values of a^2R and Sr were computed for every 20° from $\theta = 0$ to $\theta = 180°$, as follows:

θ	a^2R	Sr
0°	— 14.815	0.
20	— 3.547	+ 2.170
40	— 1.926	+ 0.340
60	— 1.501	— 0.134
80	— 1.130	— 0.623
100	— 0.438	— 1.182
120	+ 0.414	— 1.231
140	+ 0.975	— 0.699
160	+ 1.126	— 0.220
180	+ 1.133	0.

Whence

$$\begin{aligned} a^2R = &- 1.423 - 3.480\cos\theta - 1.538\cos 2\theta - 1.466\cos 3\theta \\ &- 1.551\cos 4\theta - 1.322\cos 5\theta - 1.213\cos 6\theta \\ &- 1.149\cos 7\theta - 1.116\cos 8\theta - 0.553\cos 9\theta, \\ Sr = &- 0.561\sin\theta + 0.800\sin 2\theta + 0.654\sin 3\theta \\ &+ 0.311\sin 4\theta + 0.459\sin 5\theta + 0.368\sin 6\theta \\ &+ 0.232\sin 7\theta + 0.131\sin 8\theta, \\ \int Srd\theta = &+ 0.561\cos\theta - 0.400\cos 2\theta - 0.218\cos 3\theta \\ &- 0.078\cos 4\theta - 0.092\cos 5\theta - 0.061\cos 6\theta \\ &- 0.033\cos 7\theta - 0.016\cos 8\theta. \end{aligned}$$

Substituting in $a^3r^{-3} = 1 - 3\sigma + 6\sigma^2 - 10\sigma^3 + \ldots$

the values of σ, σ^2, etc., derived from (7), (8), and (13), employing Tisserand's values of a_1, a_2, a_3, etc., we have

$$\begin{aligned} a^3r^{-3} = 1.031 &+ 0.001\cos\theta + 0.004\cos 2\theta - 0.308\cos 3\theta \\ &- 0.002\cos 4\theta - 0.001\cos 5\theta + 0.031\cos 6\theta \\ &- 0.003\cos 9\theta. \end{aligned}$$

Whence (10) gives

$$\begin{aligned} a^2P = &- 1.241 + 0.353\cos\theta - 4.399\cos 2\theta - 2.751\cos 3\theta \\ &- 2.545\cos 4\theta - 1.457\cos 5\theta - 1.386\cos 6\theta \\ &- 1.224\cos 7\theta - 1.172\cos 8\theta - 0.518\cos 9\theta. \end{aligned}$$

Differentiating (1) and comparing with (3), we have

$$\begin{aligned} n_1 &= + 0.0010, \\ n_2 &= + 0.0025, \\ n_3 &= - 0.1993, \\ n_4 &= - 0.0012. \end{aligned}$$

From equations (17) to (24) were next obtained the values given in the first column of the following table. With these and the values of P_i and S_i already

employed new values were obtained, which are given in the second column of the table:

	I	II
ν	0.014931	0.014091
log μ	9.054927	9.055352
$1/m'$	1290.	1370
a_1	− 0.0013	− 0.0012
a_2	− 0.0076	− 0.0071
a_3	+ 0.1000	+ 0.1000
a_4	+ 0.0027	+ 0.0025
a_5	+ 0.0018	+ 0.0017
a_6	+ 0.0003	+ 0.0003
a_7	0.0	0.0
a_8	+ 0.0002	+ 0.0002
n_1	+ 0.0027	+ 0.0025
n_2	+ 0.0137	+ 0.0129
n_3	− 0.1990	− 0.1990
n_4	− 0.0062	− 0.0058
n_5	− 0.0023	− 0.0022
n_6	− 0.0008	− 0.0008
n_7	+ 0.0002	+ 0.0002
n_8	− 0.0003	− 0.0003

The corresponding formulæ for the radius vector and longitude of Hyperion are

$$\frac{r}{a} = 1 - 0.0012\cos\theta - 0.0071\cos 2\theta + 0.1000\cos 3\theta + 0.0025\cos 4\theta + 0.0017\cos 5\theta + 0.0003\cos 6\theta + 0.0002\cos 8\theta, \tag{25}$$

$$w = l + 26'\sin\theta + 66'\sin 2\theta - 682'\sin 3\theta - 15'\sin 4\theta - 5'\sin 5\theta - 1'\sin 6\theta. \tag{26}$$

Finally from the formulæ

$$k = \frac{n'}{\sqrt{(1+m')}}, \quad a^3 = \frac{\mu k^2}{(n'-n)^2}, \tag{27}$$

and the values of μ and m' obtained from the second column of the above table, we have

$$\log k = 1.353507, \quad \log a = 0.085727.$$

12. As a check, the computations were carried through a second time, this time dividing the circle into 72 parts, and working independently of Tisserand's values of the smaller coefficients in the expressions for r/a and dw/dt. In computing the forces, I have assumed

$$r = a(1 + 0.1 \cos 3\theta),$$

$$w' - w = \theta + 683' \sin 3\theta.$$

To find a, if we neglect small terms of the sixth order and those containing m', we have

$$n_3 = -\frac{2a_3}{1 + \frac{3}{4}a_3^2}, \tag{28}$$

$$\nu = \frac{\frac{3}{2}a_3^2}{1 + \frac{1}{2}a_3^2}, \tag{29}$$

$$\alpha^3 = \frac{a'^3}{a^3} = \frac{n^2}{n'^2} \cdot \frac{(1 + \frac{1}{2}a_3^2 + a_3 n_3)^2}{1-\nu}; \tag{30}$$

whence

$$n_3 = 0.1485, \qquad \nu = 0.01493,$$

$$a' = 1.00000, \quad \log a = 0.08569.$$

Including only those terms which will be needed in computing m', we have

$$a^2 R = -1.280 - 1.141 \cos 3\theta - \ldots,$$

$$\int S r d\theta = -0.206 \cos 3\theta - 0.083 \cos 6\theta - 0.038 \cos 9\theta - \ldots,$$

$$a^3 r^{-3} = \quad 1.031 - 0.308 \cos 3\theta + 0.031 \cos 6\theta - \ldots,$$

$$a^2 P = -1.061 - 2.600 \cos 3\theta - \ldots.$$

Finally, the equation

$$[3a_3 P_0 + (1 + \tfrac{1}{2}a_3^2) P_3]\, m' = \left[\frac{1}{\sqrt{(1-a_3^2)}} - 9\mu(1 + \tfrac{1}{2}a_3^2)\right] a_3, \tag{31}$$

obtained by substituting (17) in (19) and neglecting small terms of the sixth order, gave

$$m' = \tfrac{1}{13\,79}.$$

13. This approximate value for the mass of Titan is about eight times that given by Tisserand. The reasons for this are apparent: To terms of the first order, $\mu = (n' - n)^2 / n^2$ and $m'P_3 = (1 - 9\mu)\, a_3$, P_3 being obtained from the hypothesis $r = a$ and $r' = a'$. $m'P_3$, however, is so small that the introduction of a more accurate value of μ and the addition of the remaining terms given in equation (19) change it ($m'P_3$) from $-$ 0.0006 to $-$ 0.0021, while P_3 changes from $-$ 6.54 to $-$ 2.60. The disturbance produced by Titan is, of course, greatest near conjunction; but Hyperion is then always at apo-saturnium, and accordingly the average distance between the two satellites is approximately $a\,(1 + a_3) - a'$ instead of $a - a'$; i. e. to terms of the first order, 0.275a instead of 0.175a.

14. Prof. Newcomb in a discussion of the motion of the peri-saturnium of Hyperion*, assuming, as I have done, that the orbit of Titan is a circle, has found $m' = \frac{1}{12800}$; but in so doing has assumed n to be the *elliptic* mean motion, instead of the mean motion in *longitude:* whereas the mean value of the mean motion in the variable ellipse is, approximately, $k / a^{\frac{3}{2}}$. Assuming

$$k = n' \quad \text{and} \quad \mathrm{n} = \frac{k}{a^{\frac{3}{2}}} = 16^\circ.7931,$$

we have $$4\mathrm{n} - 3n' = -0^\circ.5585;$$

whence, using the same values for the forces as those employed by Newcomb,

$$m' = \frac{4\mathrm{n} - 3n'}{40.6\,\mathrm{n}} = \frac{1}{1209},$$

which agrees approximately with the values already found.

*On the Motion of Hyperion. A New Case in Celestial Mechanics, pp. 365 7.

[TO BE CONTINUED.]

ON THE MOST PROBABLE VALUE OF THE LATITUDE, AND ITS THEORETICAL WEIGHT, FROM ENTANGLED OBSERVATIONS OCCURRING IN THE USE OF TALCOTT'S METHOD.

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Each observation of a pair of stars gives an equation of the well-known form,

$$\varphi = \tfrac{1}{2}(\delta_s + \delta_n) + \tfrac{1}{2}(M_s - M_n)R + \tfrac{1}{2}(l_s + l_n) + \tfrac{1}{2}(r_s - r_n) + \tfrac{1}{2}(\zeta_s + \zeta_n); \quad (1)$$

where the terms of the right-hand member depend respectively on the star-declinations, the micrometer readings, the level readings, the refractions, and the reductions to the meridian.

The entanglement consists in the use of the same star in more than one pair. One or more stars on one side of the zenith may be combined with one or more stars on the other side of the zenith. For the sake of uniformity in the reductions and of ready detection of errors in the star-places or instrumental readings, it is best to combine the stars in simple pairs. Unless then, with a given setting of the level, the number of stars observed on each side of the zenith is the same, some star, or stars, on one side must be combined, individually, with more than one star on the other side of the zenith.

Suppose the star whose declination is δ_A to be combined with the stars on the other side of the zenith, whose declinations are δ_1, δ_2, δ_3, ... Let these symbols denote the mean declinations of the stars.

Since the reductions to apparent place for the several dates do not come into consideration for the purpose of this paper, we may, for brevity, represent any functions of them whatever that may occur in our preliminary equations by the symbols Δ_A, Δ_1, Δ_2, Δ_3, . . . for the corresponding stars, as indicated by the suffixes.

Let

I_A, I_1, I_2, I_3, . . . = values, for the corresponding stars as indicated by the suffixes, of the right hand member of equation (1) exclusive of the first term,

n_1, n_2, n_3, . . . = number of observations of the pairs into which δ_1, δ_2, δ_3, . . . enter,

ν = total number of nights on which observations were made of any pairs of the group,

μ = number of pairs in the group,

m_1, m_2, m_3, . . . = number of pairs of the group observed on the 1st, 2nd, 3rd, etc., nights,

$\varphi_1, \varphi_2, \varphi_3, \ldots$ = arithmetical means of the values of the latitude resulting from the individual observations of the 1st, 2nd, 3rd, etc., pairs,

φ_0 = most probable value of the latitude from the entangled group of pairs under consideration,

$p_1, p_2, p_3, \ldots$ = weights with which $\varphi_1, \varphi_2, \varphi_3, \ldots$ are to be combined to produce φ_0,

$P = p_1 + p_2 + p_3 + \ldots = \Sigma p$,

W_0 = theoretical weight of φ_0,

$v_1, v_2, v_3, \ldots$ = residual corrections of $\varphi_1, \varphi_2, \varphi_3 \ldots$ with reference to φ_0,

r = probable error of a latitude in general,

r_0 = probable error of φ_0,

r_δ = probable error of a star-declination,

r_I = probable error of a single observation of a single star with the zenith telescope = probable error of any one of the quantities I_A, I_1, I_2, I_3, etc.,

$$h = r_I^2 / r_\delta^2. \tag{2}$$

With a good instrument and a good observer we may assume $h = 2$, or $r_I^2 = 2r_\delta^2$. With adopted declinations depending on three or more good authorities, which condition can now be fulfilled generally, we may assume r_δ to have a common value, about $\pm 0''.50$, for all the stars.

1. From the successive pairs of stars we have

$$\begin{aligned} \varphi_1 &= \tfrac{1}{2}(\delta_A + \delta_1) + \Delta_A + \Delta_1 + \frac{1}{2n_1}(\Sigma_1 I_A - \Sigma I_1), \\ \varphi_2 &= \tfrac{1}{2}(\delta_A + \delta_2) + \Delta_A + \Delta_2 + \frac{1}{2n_2}(\Sigma_2 I_A - \Sigma I_2), \end{aligned} \tag{3}$$

where the meaning of the signs of summation is quite evident; and, for the determination of φ_0, the equations

$$\begin{aligned} \varphi_0 - \varphi_1 &= v_1, \\ \varphi_0 - \varphi_2 &= v_2, \\ &\ldots\ldots \end{aligned}$$

to which proper weights $p_1, p_2, \ldots$ are to be assigned. Whatever these weights may be we shall have

$$\varphi_0 = \frac{\Sigma(p\varphi)}{P}. \tag{4}$$

Substituting in (4) the values of $\varphi_1, \varphi_2, \varphi_3, \ldots$ from (3), we have

$$\varphi_0 = \tfrac{1}{2}\delta_A + \Delta_A + \frac{1}{2P}\sum_{i=1}^{i=\mu}(p_i\delta_i) + \Sigma_i\Delta_i$$

$$+\frac{1}{2P}\left[\left(\sum_1^{(1)}\frac{p_i}{n_i}\right)I_A^{(1)}+\left(\sum_1^{(2)}\frac{p_i}{n_i}\right)I_A^{(2)}+\ .\ .\ .\ +\left(\sum_1^{(\nu)}\frac{p_i}{n_i}\right)I_A^{(\nu)}\right]$$

$$-\frac{1}{2P}\left[\frac{p_1}{n_1}\sum_1^{n_1}I_1+\frac{p_2}{n_2}\sum_1^{n_2}I_2+\ .\ .\ .\ +\frac{p_\mu}{n_\mu}\sum_1^{n_\mu}I_\mu\right]. \qquad (5)$$

In the above equation the summations, except within the last brackets, are with reference to pairs of stars, the suffix i indicating the quantities that change from pair to pair. The term $\sum_{i=1}^{i=\mu}(p_i\delta_i)$ is the only one that necessarily involves all the pairs. The parenthetical exponents (1), (2), . . . , within the first brackets, indicate that the quantities to which they are attached pertain to the 1st, 2nd, etc., nights. Within the last brackets the summations are with reference to nights of observation, as in (3).

So far as the declinations and micrometer readings are concerned, each term of (5) is independent of every other term. Hence, proceeding to probable errors, we have

$$4r_0^2=\left(1+\frac{1}{P^2}\sum_{i=1}^{i=\mu}p_i^2\right)r_\delta^2$$

$$+\frac{1}{P^2}\left[\left(\sum_1^{(1)}\frac{p_i}{n_i}\right)^2+\left(\sum_1^{(2)}\frac{p_i}{n_i}\right)^2+\ .\ .\ .\ +\left(\sum_1^{(\nu)}\frac{p_i}{n_i}\right)^2\right]r_I^2$$

$$+\frac{1}{P^2}\left(\sum_{i=1}^{i=\mu}\frac{p_i^2}{n_i}\right)r_I^2,$$

where the only quantities that necessarily involve all the pairs are $\sum_{i=1}^{i=\mu}p_i^2$ and $\sum_{i=1}^{i=\mu}p_i^2/n_i$.

2. We have now to determine p_1, p_2, p_3 . . . under the condition that the resulting value of r_0 shall be a minimum.* It is evident that the maximum value of r_0 is the probable error of the value of φ from that pair which has the smallest number of observations; that is, the value of r_0 corresponding to the system of weights where the weight of that pair $=1$ and the weight of every other pair $=0$. We shall determine the minimum, therefore, if we differentiate the expression for $4r_0^2$ with reference to p_1, p_2, p_3, . . . respectively.

If, for brevity, we write equation (6) in the form

$$4r_0^2=r_\delta^2+\frac{U}{P^2},$$

**On the Algebraical and Numerical Theory of Errors of Observations and the Combination of Observations.* George Biddell Airy. §10, ¶64, and §11, ¶69.

and differentiate with reference to $p_1, p_2, p_3, \ldots$, we have a series of equations, μ in number, of the form

$$\frac{dU}{dp} = F,$$

for the determination of $p_1, p_2, p_3, \ldots$; where F is a certain function common to all the μ equations, since P is a symmetrical function of $p_1, p_2, p_3, \ldots$. Writing dU/dp in full, we have for the weight p_t of any particular pair observed on n_t nights, the equation

$$p_t r_\delta^2 + \left(\Sigma_i^{(1)} \frac{p_i}{n_i} + \Sigma_i^{(2)} \frac{p_i}{n_i} + \ldots + \Sigma_i^{(\nu)} \frac{p_i}{n_i} + p_t \right) \frac{r_t^2}{n_t} = \frac{F}{2};$$

in which the terms corresponding to any nights on which the given pair may not have been observed, are equal to zero. Substituting hr_δ^2 for r_t^2, from (2), we have μ equations as follows, for the determination of $p_1, p_2, p_3, \ldots$;

$$\frac{n_1 + h}{n_1} \cdot p_1 + \frac{h}{n_1} \left(\Sigma_i^{(1)} \frac{p_i}{n_i} + \Sigma_i^{(2)} \frac{p_i}{n_i} + \ldots + \Sigma_i^{(\nu)} \frac{p_i}{n_i} \right) = \frac{F}{2r_\delta^2} = F',$$

$$\frac{n_2 + h}{n_2} \cdot p_2 + \frac{h}{n_2} \left(\Sigma_i^{(1)} \frac{p_i}{n_i} + \Sigma_i^{(2)} \frac{p_i}{n_i} + \ldots + \Sigma_i^{(\nu)} \frac{p_i}{n_i} \right) = \frac{F}{2r_\delta^2} = F', \qquad (7)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

From the solution of the equations (7) we have

$$p_1 = k_1 F', \qquad p_2 = k_2 F', \qquad \ldots, \qquad (8)$$

where $k_1, k_2, k_3, \ldots$ are functions of known quantities. But, since we are concerned only with relative weights within the group of pairs, we may write

$$p_1 = k_1, \qquad p_2 = k_2, \qquad \ldots \qquad (9)$$

3. A common case in practice is where certain pairs of the group are observed on each of ρ nights only, while the remaining pairs of the group are observed on each of ν nights, including the same ρ nights. Let $\lambda =$ the number of pairs observed on ρ nights only, and let their combination-weights be represented by $p_1, p_2, p_3, \ldots p_\lambda$. Then the weights of the $\mu - \lambda$ remaining pairs will be represented by $p_{\lambda+1}, p_{\lambda+2}, \ldots p_\mu$. Evidently

$$p_1 = p_2 = \ldots = p_\lambda,$$

and

$$p_{\lambda+1} = p_{\lambda+2} = \ldots = p_\mu,$$

Let

$$s = \sum_{i=1}^{i=\mu} \frac{p_i}{n_i} = \lambda \cdot \frac{p_\lambda}{\rho} + (\mu - \lambda) \cdot \frac{p_\mu}{\nu}.$$

From (7) we have

$$(\rho + h)\, p_\lambda + h\rho s = \rho F',$$

$$(\nu + h)\, p_\mu + h\left[\rho s + (\nu - \rho)\left(s - \lambda \frac{p_\lambda}{\rho}\right)\right] = \nu F'.$$

The second of these equations reduces to

$$(\nu + h)\, p_\mu + h\nu s - h\lambda p_\lambda \left(\frac{\nu - \rho}{\rho}\right) = \nu F'.$$

Eliminating F' between these two equations, we have

$$[\nu(\rho + h) + h\lambda(\nu - \rho)]\, p_\lambda - \rho(\nu + h)\, p_\mu = 0.$$

Whence

$$\frac{p_\mu}{p_\lambda} = \frac{\nu(\rho + h) + h\lambda(\nu - \rho)}{\rho(\nu + h)}. \qquad (10)$$

The disappearance of μ in this expression is to be noted.

4. Instead of combining the values of φ resulting from the separate pairs of stars and then determining the relative weights of pairs, we might have combined the values of φ resulting from the separate nights and then have determined, according to the same principles, the relative weights of nights. It may be of interest to compare the expression thus derived corresponding to (10). If we let p_ρ and p_ν = the weight of any one of the ρ and of the $\nu - \rho$ nights respectively, we shall find, after the necessary reductions,

$$\frac{p_\rho}{p_\nu} = \frac{h\mu(\mu - \lambda + 1) + \lambda(\nu - \rho)}{h(\mu - \lambda)(\mu + 1)}.$$

For a symmetrical group of observations, under this case, we have $\lambda = \nu - \rho$, $\mu = \nu$, and, if we assume $h = 1$, the expressions for p_μ/p_λ and p_ρ/p_ν should become identical. We shall find

$$\frac{p_\mu}{p_\lambda} = \frac{\mu}{\mu - \lambda} - \frac{\lambda}{\mu + 1} = \frac{p_\rho}{p_\nu} = \frac{\nu}{\rho} - \frac{\nu - \rho}{\nu + 1}.$$

5. If we put

$$S = \left(\sum_i^{(1)} \frac{p_i}{n_i}\right)^2 + \left(\sum_i^{(2)} \frac{p_i}{n_i}\right)^2 + \ldots + \left(\sum_i^{(\rho)} \frac{p_i}{n_i}\right)^2 + \sum_{i=1}^{i=\mu} \left(\frac{p_i^2}{n_i}\right),$$

and substitute hr_δ^2 for r_I^2, equation (6) becomes

$$4r_0^2 = \left(1 + \frac{1}{P^2} \sum_{i=1}^{i=\mu} p_i^2 + \frac{h}{P^2} S\right) r_\delta^2. \qquad (11)$$

Then we have
$$W_0 = \frac{1}{r_0^2} = \frac{4P^2}{P^2 + \sum_{i=1}^{i=\mu}{}' p_i^2 + hS} \cdot \frac{1}{r_\delta^2}. \qquad (12)$$

If we wish to give the weight unity to a single observation of a single independent pair, we must assume $r_\delta^2 = \frac{2}{3}$; whence by (12)
$$W_0 = \frac{6P^2}{P^2 + \sum_{i=1}^{i=\mu}{}' p_i^2 + hS}. \qquad (13)$$

If all the pairs are observed on each of all the nights, evidently $p_1 = p_2 = p_3 = \ldots$, and (12) and (13) become respectively
$$W_0 = \frac{4\mu\nu}{(\mu+1)(\nu+h)} \cdot \frac{1}{r_\delta^2}, * \qquad (14)$$

and
$$W_0 = \frac{6\mu\nu}{(\mu+1)(\nu+h)}.$$

6. At some stations it will happen that a certain star is so near the zenith that it may be observed in both positions of the instrument, and forms a pair by itself. It may be well to present the principal formulæ above in the shape which they assume for this case. From (1) we have for a single such pair,
$$r^2 = r_\delta^2 + \frac{1}{2n} \cdot r_I^2 = \frac{2n+h}{2n} \cdot r_\delta^2;$$

and
$$r^2 = \frac{2n+h}{3n}, \quad \text{if} \quad r_\delta^2 = \tfrac{2}{3}.$$

If we suppose the stars whose declinations are denoted by δ_A and δ_1 to be identical, we have corresponding respectively to (11), (12), (13), (14), and (15) the following:—
$$4r_0^2 = \left[\left(\frac{P+p_1}{P} \right)^2 + \frac{1}{P^2} \sum_{i=2}^{i=\mu}{}' p_i^2 + \frac{h}{P^2} S \right] r_\delta^2, \qquad (11')$$

$$W_0 = \frac{4P^2}{(P+p_1)^2 + \sum_{i=2}^{i=\mu}{}' p_i^2 + hS} \cdot \frac{1}{r_\delta^2}, \qquad (12')$$

$$W_0 = \frac{6P^2}{(P+p_1)^2 + \sum_{i=2}^{i=\mu}{}' p_i^2 + hS}, \qquad (13')$$

*Sir George B. Airy, in discussing a similar problem, that of determining the Weights to be given to the Separate Results for Terrestrial Longitudes, determined by the Observation of Transits of the Moon and Fixed Stars, has derived a formula which is the equivalent of (14). (Memoirs of the Royal Astronomical Society. Vol. XIX. p. 224).

$$W_0 = \frac{4\mu\nu}{(\mu+1)(\nu+h)+2\nu} \cdot \frac{1}{r_\delta^2}, \qquad (14')$$

$$W_0 = \frac{6\mu\nu}{(\mu+1)(\nu+h)+2\nu}. \qquad (15')$$

7. From (15) we have the following:—

Table of Weights when All the Pairs are Observed on Each of All the Nights.

μ = No. of Pairs.	ν = No. of Nights.				
	1	2	3	4	5
1	1.00	1.50	1.80	2.00	2.14
2	1.33	2.00	2.40	2.66	2.86
3	1.50	2.25	2.70	3.00	3.22
4	1.60	2.40	2.88	3.20	3.43
5	1.67	2.50	3.00	3.34	3.56

It is evident from the above table that in this case the weight of the value of the latitude from the group is increased by observations on additional nights rather than by observations on additional pairs; which is contrary to the usual case, that of independent pairs.

8. In practice the rigorous method of determining the values of p_1, p_2, p_3, . . . by (9), might sometimes occasion an undue amount of labor. It is not important oftentimes to determine theoretical weights rigorously, since, in general, a considerable change in the independent variable makes an inconsiderable change in the value of the function at or near a maximum or minimum. Sir George B. Airy shows, in his work on the Theory of Errors, already referred to by foot-note, that, given two independent measures whose combination-weights are as 4 : 1, any combination-weights from 2 : 1 up to 16 : 1 may be used without increasing the probable error of the result more than one-fifteenth part of itself. Still, a good computer would wish to use the best weights available, exercising his judgment as to the relative importance of a certain theoretical accuracy in the result and the labor necessary to attain theoretically that accuracy.

Sir George B. Airy, in his memoir referred to by foot-note under (14), after deriving, for a particular example, a set of equations corresponding to (9), recommends in general the use of certain approximate weights. His approximate

weight of any one observation, for the kind of observations we are considering, is the weight which that observation would have if it were assigned its equal proportionate part of the weight which the final value of the latitude from the entire group of observations would have, if, first, the total number of pairs observed were the same as the number of pairs on the night in question; and if, second, the total number of nights were the same as the number of nights on which the pair in question was observed; and if, third, all the pairs were observed on each of all the nights. This approximate weight is found, therefore, from (14) by dividing by $\mu\nu$, the number of observations, and substituting in the denominator for μ and ν, the corresponding m and n as described above. This gives

$$p = \frac{c}{(m+1)(n+h)}, \tag{16}$$

where c is an arbitrary constant determining the scale of the weights, and p is the weight of a single observation of a pair of stars.

9. For the case under which (10) was derived, a comparison is readily made of the ratio of the approximate weights from (16), with the ratio of the true weights from (10), and thus a better idea can be gained of their relations in general. From (16), disregarding the constant in the numerator, for this case we have

$$p_\lambda = \frac{\rho}{(\mu+1)(\rho+h)},$$

$$p_\mu = \frac{\rho}{(\mu+1)(\nu+h)} + \frac{\nu-\rho}{(\mu-\lambda+1)(\nu+h)}.$$

Whence

$$\frac{p_\mu}{p_\lambda} = \frac{\rho+h}{\nu+h}\left[1 + \frac{(\nu-\rho)(\mu+1)}{(\mu-\lambda+1)\rho}\right]. \tag{17}$$

If we let f_1 and f_2 equal the values of p_μ / p_λ from (10) and (17), respectively, we have

$$f_1 = \frac{\nu(\rho+h)}{\rho(\nu+h)} + \frac{h\lambda(\nu-\rho)}{\rho(\nu+h)},$$

$$f_2 = \frac{\nu(\rho+h)}{\rho(\nu+h)} \cdot \frac{\mu - \frac{\rho}{\nu}\lambda + 1}{\mu-\lambda+1};$$

or, transforming these expressions,

$$f_1 = \frac{\nu}{\nu+h} \cdot \frac{\rho+h}{\rho} \cdot \left[1 + \frac{\lambda(\nu-\rho)}{\nu} \cdot \frac{h}{\rho+h}\right], \tag{18}$$

$$f_2 = \frac{\nu}{\nu+h} \cdot \frac{\rho+h}{\rho} \cdot \left[1 + \frac{\lambda(\nu-\rho)}{\nu} \cdot \frac{1}{\mu-\lambda+1}\right]. \tag{19}$$

It is evident from these expressions that both f_1 and f_2 increase continually

as λ increases, and diminish continually as ρ increases. Hence each attains its greatest value when we have $\lambda = \mu - 1, \rho = 1$. For $\rho = \nu$ each becomes unity.

Subtracting (19) from (18) and reducing, we have, for the correction to the approximate ratio f_2,

$$f_1 - f_2 = \frac{1}{\nu + h} \cdot \frac{\nu - \rho}{\rho} \cdot \lambda \cdot \frac{h(\mu - \lambda) - \rho}{\mu - \lambda + 1}. \qquad (20)$$

From this expression we have, evidently,

$$f_1 - f_2 \gtreqless 0 \quad \text{according as} \quad \rho \lesseqgtr h(\mu - \lambda);$$

and that $f_1 - f_2$ diminishes continually, in algebraic magnitude, as ρ increases. If we differentiate the factor $\lambda \cdot \frac{h(\mu - \lambda) - \rho}{\mu - \lambda + 1}$ with reference to λ, and put the result equal to zero, we find the values of λ that give a maximum or minimum of $f_1 - f_2$. The result, after reducing, is a quadratic equation; whence

$$\lambda = (\mu + 1) \pm \sqrt{\left[(\mu + 1) \cdot \frac{\rho + h}{h}\right]}.$$

It is evident from (20) that $f_1 - f_2$, for any given value of ρ, passes from positive to negative values as λ is increasing. Therefore the smaller value of λ above corresponds to the maximum of $f_1 - f_2$. Since λ cannot be greater than μ, the larger value of λ above, corresponding to the true minimum of $f_1 - f_2$, is inadmissible; but we have a least value of $f_1 - f_2$, algebraically, when $\lambda = \mu - 1$; although for $\lambda = \mu$, when also ρ must equal ν, we have $f_1 - f_2 = 0$. Hence we have the maximum value of $f_1 - f_2$ when $\lambda = (\mu + 1) - \sqrt{\left[(\mu + 1)\frac{\rho + h}{h}\right]}$, $\rho = 1$, and the least value when $\lambda = \mu - 1, \rho = \nu - 1$.

To compare numerical results, let us suppose the simple example where $\mu = 5, \nu = 5, h = 2$. By the above we have the maximum of $f_1 - f_2$ when $\lambda = 3, \rho = 1$. Substituting these numerical values in (20), we have $f_1 - f_2 = \frac{12}{7}$. Substituting also in (18) and (19), we have $f_1 = \frac{39}{7}, f_2 = \frac{27}{7}$; which are consistent with the value of $f_1 - f_2$. Hence f_2 is in error here by nearly one-third of the true value of the ratio of the weights. We have the least value of $f_1 - f_2$ when $\lambda = 4, \rho = 4$. Again substituting in (20), $f_1 - f_2 = -\frac{1}{7}$. The corresponding values of f_1 and f_2, from (18) and (19) are $\frac{19}{14}$ and $\frac{21}{14}$; so that here f_2 is in error only about one-tenth part. For $\mu = 23$, a higher number than would occur, of course, in latitude work, and $\nu = 5$, we should find for the maximum, $f_1 - f_2 = \frac{108}{7}$, while $f_1 = \frac{159}{7}$; whence f_2 would be in error by $\frac{108}{159}$ or 0.68 times the true value.

The numerical data, at least in the case of observations for latitude by Talcott's method, are so few and simple in any one example, it would seem more satisfactory to make a thorough solution for the weights, employing equation

(10) for the special case under which it was derived, than to depend on any system of approximate weights.

10. Mr. C. A. Schott* and Prof. T. H. Safford† have discussed, to some extent, the question of the final weight in the case of entangled observations for latitude by Talcott's method. Both writers consider the general case where N stars north of the zenith are observed with S stars south of the zenith with only one reversal of the instrument. Each, disregarding the error of observation, that is, taking into account the entanglement by declinations only, gives the readily derived expression $\frac{2NS}{N+S}$ as the factor by which the ordinary weight of the group, considered as one pair, is to be multiplied, before combining with values of the latitude from other pairs or groups of stars. In the case of such groups as we have under consideration, either N or S is unity, and the factor is of the form $\frac{2S}{1+S}$. Professor Safford gives a table of weights which, reduced to accord with the assumption of the weight unity for a single observation on a single independent pair, is as follows:

Number of Stars $= N + S$.	Number of observations of group.	Weight of mean value of latitude from group.
2	1	1.00
	2	1.50
	3 to 6	2.00
3	3 to 6	2.67
4		4.00
5		4.80

For a less number of observations than 3, the last three weights of the table are to be diminished in the proportion of the first three. It will be seen that the weights of the above table rise to a value exceeding one and one-half times the corresponding weights of the table computed by (15).

11. In computing the probable error of the final value of the latitude from

*Report of the Superintendent of the U. S. Coast and Geodetic Survey for 1880. Appendix No. 14, p. 255.

†Report of the Secretary of War 1878–9. Report of Capt. George M. Wheeler, Corps of Engineers, U. S. A. Surveys West of the One Hundredth Meridian, pp. 1987, 1995.

all the independent pairs and groups observed, by the formula

$$r = 0.6745 \sqrt{\left[\frac{\Sigma(pvv)}{(m-1)\Sigma p}\right]},$$

a group of entangled pairs should be represented by a single weight, $p = W_0$, and a single residual, v, the difference between φ_0 and the general mean from all the independent pairs. But if, in computing the final value of the latitude, it is desired for the sake of uniformity, to enter the values of the latitude from the individual pairs of the group, the arithmetical means of the individual observations on the several pairs may be multiplied respectively by the combination-weights, $p_1' = W_0 / P \cdot p_1$, $p_2' = W_0 / P \cdot p_2$, . . . ;

since $\sum_{i=1}^{i=\mu} p_i' = W_0$ and $\sum_{i=1}^{i=\mu} (p_i' \varphi_i) = W_0 \varphi_0$.

12. In some cases the same star will enter into more than one-group of entangled pairs. The value of the latitude is then to be computed from each group, by (4), and the values from the different groups combined by a process similar to that already employed in combining the individual pairs of a single group; as illustrated in the example given below.

13. The following observations, whose reduction will illustrate the application of the formulæ above, are taken from the Report upon United States Geographical Surveys West of the One Hundredth Meridian, in charge of First Lieutenant Geo. M. Wheeler, Corps of Engineers, U. S. Army; Vol. II. pp. 68-9. This example is presented not as one of a kind that frequently occur, or that would reward adequately the labor necessary for a thorough reduction, but as showing to an interesting extent the application of the above principles; which are applicable also to other kinds of observations than those for latitude.

Station: Beaver, Utah. Latitude = + 38° 16′.

Designation of Pairs.	Stars of Pair.		Observed Values of Latitude.		Mean = ϕ.
	South.	North.	1872. August 23.	1872. August 24.	
(1)	B. A. C. 7733	B. A. C. 7754	25″.3	〃	25″.3
I	7807	7754	——	23.5	23.5
II	7807	7778	22.2	22.9	22.55
III	7807	7782	23.0	22.6	22.80
IV	7807	7779	——	23.2	23.2

We shall assign the weight 1 to a single observation of a single independent pair, in accordance with the assumptions $h = 2, r_\delta^2 = \frac{2}{3}$.

If we treat the pairs as independent and weight them by the ordinary formula $p = \frac{2n}{nr_\delta^2 + r_I^2}$, we have $\varphi_0 = 20'' + \frac{20''.03}{6} = 23''.34$; and the weight of φ_0 will be 6.00, the sum of the individual weights.

Since, in the example, the number of observations on any one of the pairs is so small, the weights of the table on p. 181 would approximate closely to the true theoretical weight. If we assume $n = 1\frac{1}{2}$ as a sort of average of the number of observations on each pair of stars, we have, from the system on which that table rests, 3.43 as the weight of the final value of the latitude.

We have here the pair (1) entangled with the group I, II, III, IV; so that we must first determine the most probable value of the latitude from the group, and then combine it suitably with the value from pair (1). For the group, which evidently comes under the case for which (10) was derived, we have $\mu = 4$, $\nu = 2$, $\lambda = 2$, $\rho = 1$, and we will assume throughout $h = 2$. By (10) we have

$$\frac{p_\mu}{p_\lambda} = \frac{2 \times 3 + 2 \times 2}{4} = \tfrac{5}{2}.$$

We have the following as the computation of φ_0 and W_0:

Pair.	ϕ	p	$p\phi$	p^2	p^2/n	p/n	
						Aug. 23.	Aug. 24.
I	23".5	2	47".0	4	4	—	2
II	22.55	5	112.8	25	$\frac{25}{2}$	$\frac{5}{2}$	$\frac{5}{2}$
III	22.80	5	114.0	25	$\frac{25}{2}$	$\frac{5}{2}$	$\frac{5}{2}$
IV	23.2	2	46.4	4	4	—	2
Sums		14	320.2	58	33	5	9

$$F^2 = 196 \cdot \left(\Sigma_i^{(1)} \frac{p_i}{n_i} \right)^2 = 25 \cdot \left(\Sigma_i^{(2)} \frac{p_i}{n_i} \right)^2 = 81,$$

$$S = 25 + 81 + 33 = 139,$$

and, by (15), $\varphi_0 = \frac{320.2}{14} = 22''.87,$

$$W_0 = \frac{6 \times 196}{196 + 58 + 2 \times 139} = 2.21.$$

Next let

φ' = the value of the latitude from pair (1),

φ_0' = the most probable value of the latitude from all the pairs of the example,

W', W'' = the combination-weights of φ', φ_0 required to produce φ_0',

r_0' = the probable error of φ_0',

W_0' = the theoretical weight of φ_0',

B, A, 1, 2, 3, 4 denote that the symbols to which they are attached pertain to the stars B. A. C. 7733, 7807, and the remaining stars in the order in which they occur in the example.

From (3) we have

$$\varphi' = \tfrac{1}{2}\left[\delta_B + \Delta_B + \delta_1 + \Delta_1 + (I_B - I_1)\right].$$

From (5) we have, bearing in mind that we represent in general by the symbol Δ any function of the reduction to apparent place that may occur,

$$\varphi_0 = \tfrac{1}{2}\left[\delta_A + \Delta_A + \frac{p_1}{P}\delta_1 + \Delta_1 + \frac{1}{P}\sum_{i=2}^{i=\mu}(p_i\delta_i)\right] + \tfrac{1}{2}T;$$

where $\frac{1}{2}T$ represents all that is left of the right-hand member of (5) after taking out the five terms enclosed in the brackets of this last equation.

Substituting these values of φ' and φ_0 in the equation $\varphi_0' = \dfrac{W'\varphi' + W''\varphi_0}{W' + W''}$, we have

$$\varphi_0' = \frac{1}{2(W' + W'')}\left[W'\delta_B + W''\delta_A + \left(W' + \frac{p_1}{P}W''\right)\delta_1 + \frac{1}{P}W''\sum_{i=2}^{i=\mu}(p_i\delta_i) + \Delta_B + \Delta_A + \Delta_1\right] + \frac{1}{2(W' + W'')}\left[W'(I_B - I_1) + W''T\right].$$

From this we derive the equation for $r_0'^2$ just as (6) was derived from (5). Referring to the value of S, however, preceding (11), it appears that the square of the probable error of T is $\frac{1}{P^2}Sr_I^2$. Letting $r_I^2 = hr_\delta^2$ and combining terms with reference to W' and W'', we have

$$4r_0'^2 = \frac{1}{(W' + W'')^2}\left[2W'^2(1 + h) + \frac{2p_1}{P}W'W'' + W''^2\left(1 + \frac{p_1^2}{P^2} + \frac{1}{P^2}\sum_{i=2}^{i=\mu}p_i^2 + \frac{h}{P^2}S\right)\right]r_\delta^2.$$

The parenthesis after W''^2 reduces to $\frac{1}{P^2}(P^2 + \sum_{i=1}^{i=\mu}p_i^2 + hS)$, which, referring to (13), is the same as $6/W_0$, assuming $r_\delta^2 = \frac{2}{3}$. Hence we have

$$4r_0'^2 = \frac{1}{(W' + W'')^2}\left[2(1 + h)W'^2 + \frac{2p_1}{P}W'W'' + \frac{6}{W_0}W''^2\right]r_\delta^2 \quad (21)$$

From this expression we have to determine the values of W' and W'' that reduce $r_0'^2$ to a minimum; just as (7) was derived from (6). Differentiating, we have

$$4(1+h)W' + 2\frac{p_1}{P}W'' = F',$$

$$\frac{2p_1}{P}W' + \frac{12}{W_0}W'' = F'.$$

For computing the numerical values of the coefficients of these equations, we have $h = 2$, and from the reduction of the I, II, III, IV above, $p_1 = 2$, $P = 14$, $W_0 = 2.21$. Solving the equations we obtain $W' / W'' = 0.44$; or $W' = 0.44$, $W'' = 1$. Hence

$$\varphi_0' = 20.'' + \frac{0.44 \times 5.''3 + 2.''87}{1.44} = 23.''61;$$

and from (21), assuming $r_\delta^2 = \frac{2}{3}$, and substituting for the other letters the numbers given above, we have $W_0' = 1 / r_0'^2 = 3.11$ instead of 6.00 as by the ordinary formula. Therefore, as the final result, we have: latitude $= +\ 38°16'\ 23.''61$ with weight 3.11.

VENABLE'S MODERN GEOMETRY.*

This little manual, prepared as an appendix to Professor Venable's edition of Legendre's Elements of Geometry, is a worthy addition to that excellent text-book. The same judicious arrangement, care in selection of material, accuracy, and completeness of presentation characterize both.

The subjects treated are, in order: Transversals, and some of their applications to the geometry of position, after the methods of Carnot; anharmonic ratios, and the theorems of Pascal and Brianchon for the circle, with applications; harmonic rows and pencils, and the harmonic properties of the complete quadrilateral; poles and polars in the circle; reciprocal polars and the law of duality; radical axes; and axes and centres of similitude.

The treatment is rather metrical than descriptive, so that the learner would get from this manual a quite inadequate conception of the genuine and characteristic methods of the Modern Geometry as expounded by Steiner, von Staudt, Reye, Cremona, and their disciples; or even by Chasles. But in some respects, this alliance with the metrical methods of the Euclidean geometry is to be pre-

* Introduction to Modern Geometry. By Charles S. Venable, LL. D., Professor of Mathematics in the University of Virginia. University Publishing Company: New York, 1887.

ferred for the beginner. The change of intellectual climate is less sudden; the introduction of novel conceptions less bewildering; and the powerful methods of the modern synthesis will be better appreciated for this gradual revelation of their freedom and scope. In a somewhat careful reading no error likely to mislead the student has been found except in the statements as to reciprocal polars in 82 *seq.*, which are entirely wrong, and inconsistent with the correct statements in 77–81. The locus of Q is the *inverse* to the locus of P, not the *reciprocal polar*. And the statement of 83, that the locus in the case of a circle is a conic, is not only wrong, but contradicts the correct statement of 82.

The well-selected exercises are carefully assorted and appended to the appropriate sections, instead of being thrown confusedly to the end of the book. To each exercise are added hints for its solution; these would seem to furnish an amount of help to the learner always ample, perhaps in some cases excessive; a few difficulties might have been left unsolved. [*W. M. T.*]

SOLUTIONS OF EXERCISES.

ACKNOWLEDGMENTS.

R. D. Bohannan, 145, 146, 147, 148, 149, 150; W. W. Campbell, 152; G W. Coakley, 151, 152, 156; R. H. Graves, 147, 150, 153, 154; A. Hall, 145, 146, 148, 149, 150, 152; Ellen A. Hayes, 152; J. E. Hendricks, 156; Wm. Hoover, 152; H. A. Howe, 156; J. M. Ingalls, 156; Alfred C. Lane, 151, 155, 156. Artemas Martin, 155; J. S. Miller, 147, 148.

145

FIND the equation to the circle through the feet of the normals from (h, k) to the parabola $y^2 = 2px$.

SOLUTION.

Circle $x^2 + y^2 + 2gx + 2fy + c$ cuts parabola $y^2 - 2px$ in points whose ordinates are roots of

$$y^4 + y^2(4p^2 + 4pg) + 8p^2fy + 4p^2c = 0. \qquad (1)$$

The ordinates of the feet of the normal from (h, k) on $y^2 - 2px$ are roots of

$$y^3 + y(2p^2 - hp) - 2p^2k = 0. \qquad (2)$$

Equations (1) and (2) have three common roots if

$$2f = -\tfrac{1}{2}k, \quad 2g = -p - \tfrac{1}{2}h, \quad c = 0.$$

[*R. D. Bohannan, and others*].

146

FIND the condition that the chords AB, CD in an ellipse should meet the transverse axis in points equidistant from the centre, the points A, B, C, D being given by their eccentric anomalies α, β, γ, δ.

SOLUTION.

Chords AB, CD make on the transverse axis intercepts

$$a \cos \tfrac{1}{2}(\alpha - \beta) \sec \tfrac{1}{2}(\alpha + \beta), \quad a \cos \tfrac{1}{2}(\gamma - \delta) \sec \tfrac{1}{2}(\gamma + \delta).$$

These are equal and opposite if

$$\tan \tfrac{1}{2}\alpha \tan \tfrac{1}{2}\beta \tan \tfrac{1}{2}\gamma \tan \tfrac{1}{2}\delta = 1.$$

[*A. Hall, and others.*]

147

THE point P on an ellipse is the projection of Q on the circle on the transverse axis. $P'Q'$ are the points diametrically opposite. Find the area of the parallelogram formed by the four points in terms of the eccentric anomaly of P.

SOLUTION.

Angle $PQQ' = 90^\circ - p$; $PQ = (a - b) \sin p$; $QQ' = 2a$; hence the required area is

$$2a(a - b) \sin p \sin (90^\circ - p) = a(a - b) \sin 2p.$$

148

FIND the area of the parallelogram formed by tangents at the same points.

SOLUTION.

The tangent at P is $ay \sin p + bx \cos p - ab$. This is cut by the diameter $-x + y \tan p$, parallel to the tangent at Q, where

$$y = \frac{ab}{(a - b) \sin p}.$$

The required area is $4ay \sec p$.

149

FROM a point P whose distances from the foci of an ellipse are r, r' tangents are drawn to the curve. Find the angle between them.

SOLUTION.

Let FM, $F'M' = r, r'$ be focal radii to the points of contact. Intersect them in N, N' by circles about F', F with radius $2a$. Then the sides of PFN', $PF'N$ are r, r', $2a$, and angles $F'PN$, FPN' are equal. Therefore MPN' equals FPN' and

$$\cos MPN' = \frac{r^2 + r'^2 - 4a^2}{2rr'}.$$

[Salmon, *Conics*, p. 192.]

150

SHOW that the equation to the pair of tangents from $x'y'$ to $b^2x^2 + a^2y^2 = a^2b^2$ may be written

$$b^2(x - x')^2 + a^2(y - y')^2 = (xy' - x'y)^2.$$

SOLUTION.

Transfer the origin to $x'y'$, and express the condition that the intersections of the ellipse with $y = mx$ shall coincide. This condition is reducible to

$$a^2m^2 + b^2 = (y' - mx')^2.$$

Whence the result follows easily.

151

INTEGRATE the differential

$$\frac{\sin\theta + \cos\theta}{\sin^4\theta + \cos^4\theta}\,d\theta. \qquad [E.\ Frisby.]$$

SOLUTION.

Substitute $x = \sin\theta - \cos\theta$; we get

$$du = \frac{2dx}{1 + 2x^2 - x^4};$$

whence
$$u = \frac{\sqrt{2}}{4\alpha}\log\frac{x+\alpha}{x-\alpha} + \frac{\sqrt{2}}{2\beta}\tan^{-1}\frac{x}{\beta},$$

where $\alpha = \sqrt{[\sqrt{(2+1)}]}$, $\beta = \sqrt{[\sqrt{(2-1)}]}$. [*G. W. Coakley, and others.*]

152

IF r and r' be *radii vectores* in a parabola, and $2f$ the difference of the true anomalies, the area of the parabolic sector between the *radii* is

$$\tfrac{1}{3}\sqrt{(rr')}\sin f.[r + r' + \sqrt{(rr')}\cos f]. \qquad [A.\ Hall.]$$

SOLUTION.

The area of the segment between the chord PP', and the arc of $y^2 - 2p - x$ is

$$\frac{(y' - y)^3}{12p};$$

we have also $x' - x = r' - r$ and $\theta' - \theta = 2f$; and from the geometrical properties of the curve

$$p = 2r\cos^2\tfrac{1}{2}\theta = 2r'\cos^2\tfrac{1}{2}\theta',$$

$$r^2 - 2rr'\cos 2f + r'^2 = (r' - r)^2 + (y' - y)^2.$$

Elimination of p, y, y' gives for the area of the segment

$$\tfrac{1}{3}\sqrt{(rr')}\sin f.[r + r' - 2\sqrt{(rr')}\cos f].$$

Add to this the area of the triangle $PFP' = rr'\sin f\cos f$. [*A. Hall.*]

153

THE normals to a parabola at the points P, Q, R pass through the point O and meet the axis at the points P_1, Q_1, R_1; S is the focus and $4a$ is the length of the *latus rectum*; show that

$$PP_1 \cdot QQ_1 \cdot RR_1 = 8a^2 \cdot SO. \qquad [R.\ H.\ Graves.]$$

SOLUTION.

The length of the normal is $n = \frac{1}{2}\sqrt{(p^2 + y'^2)}$. Eliminate by the aid of this relation y' from the equation to the normal. We get the cubic in n^2

$$(n^2 + p^2 - 2px)^2 (n^2 + p^2) = 4p^4 y^2.$$

The product of the roots gives

$$n_1 n_2 n_3 = 2p^2 \sqrt{[y^2 + (x - \tfrac{1}{2}p)^2]}.$$

154

FIND the locus of the point from which the four normals drawn to an equilateral hyperbola form an harmonic pencil. [*R. H. Graves*]

SOLUTION.

The directions of the four normals from (x,y) to $x^2 - y^2 = a^2$ are found from

$$x^2 m^4 - 2xym^3 - (x^2 - y^2 - 4a^2) m^2 + 2xym - y^2 = 0.$$

These form an harmonic pencil if

$$8x^2(x^2 - y^2 - 4a^2)y^2 + 2x^2y^2(x^2 - y^2 - 4a^2) = 12x^4y^2 - 12x^2y^4 - \tfrac{2}{9}(x^2 - y^2 - 4a^2)^3$$

That is, if $(x^2 - y^2 - 4a^2)^3 = 216a^2x^2y^2$. [*R. H. Graves.*]

155

A tells the truth b times out of a; B, whose veracity is unknown, joins A in affirming the occurrence of an event. What is the probability that the event did occur? [*Artemas Martin.*]

SOLUTION.

If $c = b \div a$ is the probability of any statement by A; x that of any statement by B; the probability that they agree in the truth is cx, in an untruth is $(1 - c)(1 - x)$. The probability sought, if x were known, would be

$$p_x = \frac{cx}{cx + (1 - c)(1 - x)}$$

As x is unknown, and may have any value from 0 to 1, the required probability is

$$p = \int_0^1 p_x dx \div \int_0^1 dx = \frac{b}{2b - a}\left[1 - \frac{a - b}{2b - a} \log \frac{b}{a - b}\right].$$

[*Artemas Martin.*]

EXERCISES.

157

FIND the locus of the instantaneous centre of a tangent to a parabola when one point of the tangent moves in the tangent at the vertex.

[*De Volson Wood.*]

158

SHOW that the equation to the tangent cone from (x', y', z') to $b^2c^2x^2 + a^2c^2y^2 + a^2b^2z^2 - a^2b^2c^2 = 0$, may be written

$$b^2c^2(x-x')^2 + a^2c^2(y-y')^2 + a^2b^2(z-z')^2 = a^2(yz'-y'z)^2 + b^2(xz'-x'z)^2 + c^2(xy'-x'y)^2.$$

[*R. H. Graves.*]

159

FIND the centre of gravity of the loop of the Folium of Descartes.

[*R. H. Graves.*]

160

SHOW that the circles of curvature at the node of the Folium of Descartes pass through the middle point of the arc of the loop. [*R. H. Graves.*]

161

THE nth pedal of the curve $F(r, p) = 0$, has for its equation

$$F\left(\frac{r^{n+1}}{p^n}, \frac{r^n}{p^{n-1}}\right) = 0.$$

[*R. H. Graves.*]

162

A BODY is projected at an angle of 30° with the horizon, with a given velocity. Determine the constant resistance it must suffer in the direction contrary to its motion in order that it may come to rest when it returns to the horizontal plane whence it started. Also determine the horizontal range, time of flight, and length of trajectory. [*Jas. M. Ingalls.*]

163

AN elliptic plate of semi-axes a and b, thickness c, and density δ, is revolved at an angular velocity Ω, about a line parallel to its minor axis, and at a fixed distance from it, in a plane through this axis perpendicular to the plane of the plate. Find the straining action on the plate along the line of its minor axis; and the greatest safe value of Ω, if σ be the admissible stress for unit of area on the material of the plate. [*Jas. S. Miller.*]

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